

Bethe vectors and form factors for two-component Bose gas

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This short note presents works done in collaboration with S. Pakuliak (JINR, Dubna) and N. Slavnov (Steklov Math. Inst., Moscow). It is a summary of the articles [arXiv:1412.6037](#), [arXiv:1501.07566](#), [arXiv:1502.01966](#) and [arXiv:1503.00546](#). Here are mentioned only references used for our calculations. A detailed list of references can be found in our articles mentionned here.

1 Two-component Bose gas: general context

We consider a one-dimensional bose gas, with delta interaction and an internal degree of freedom. The continuous version is described by the Non-Linear Schrödinger (NLS) Hamiltonian

$$H_{NLS} = \int_0^L (\partial_x \Psi^\dagger \partial_x \Psi + c \Psi^\dagger (\Psi^\dagger \Psi) \Psi) dx,$$

where $\Psi(x, t) = \begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix}$ is a bosonic two-component vector satisfying the canonical commutation relations

$$[\psi_\alpha(x, t), \psi_\beta^\dagger(y, t)] = \delta_{\alpha\beta} \delta(x - y)$$

and we assume periodic boundary conditions $\Psi(x + L, t) = \Psi(x, t)$ (i.e. we are on a circle).

Aims and tools

We aim at computing the form factors of local operators. In particular for $x \in [0, L]$ and $j, k = 1, 2$ we wish to calculate $\mathbb{C} \psi_j^\dagger(x) \psi_k(x) \mathbb{B}$, $\mathbb{C} \psi_j^\dagger(x) \mathbb{B}$ and $\mathbb{C} \psi_j(x) \mathbb{B}$, that we will present below.

For such a purpose, we will be in the general context of Algebraic Bethe Ansatz (ABA). More specifically, we will use

1. A lattice version of the model and compute the Bethe vectors (BVs) through ABA
2. An (auxiliary) composite model to obtain a convenient expression for local operators
3. The zero mode method that helps to relate different form factors
4. The twisted transfer matrix method that allows to compute diagonal form factors

2 Lattice version of the Bose gas

The lattice version of the Bose gas was introduced in [10] and its ABA developed in [3]:

$$L_0(u|n) = \begin{pmatrix} 1 + \frac{c\Delta^2 \hat{N}_1(n)}{2-iu\Delta} & \frac{c\Delta^2 \psi_1^\dagger(n)\psi_2(n)}{2-iu\Delta} & \frac{-i\Delta\psi_1^\dagger(n)Q(n)}{1-iu\Delta/2} \\ \frac{c\Delta^2 \psi_2^\dagger(n)\psi_1(n)}{2-iu\Delta} & 1 + \frac{c\Delta^2 \hat{N}_2(n)}{2-iu\Delta} & \frac{-i\Delta\psi_2^\dagger(n)Q(n)}{1-iu\Delta/2} \\ \frac{i\Delta Q(n)\psi_1(n)}{1-iu\Delta/2} & \frac{i\Delta Q(n)\psi_2(n)}{1-iu\Delta/2} & \frac{2+iu\Delta}{2-iu\Delta} + \frac{c\Delta^2 \hat{\rho}(n)}{2-iu\Delta} \end{pmatrix}_0 \quad (1)$$

In the formula above, n labels the lattice site number and plays the role of x , while 0 labels the auxiliary space $V = \text{End}(\mathbb{C}^3)$. Δ is the lattice spacing and u is the spectral parameter.

$\Psi(n) = \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \end{pmatrix}$ is a discretized version of $\Psi(x, t)$ and obeys

$$[\psi_i(n), \psi_k^\dagger(m)] = \frac{1}{\Delta} \delta_{ik} \delta_{nm}.$$

We use the notation $Q(n) = \sqrt{c + \frac{c^2 \Delta^2}{4} \hat{\rho}(n)}$, where $\hat{\rho}(n) = \hat{N}_1(n) + \hat{N}_2(n)$ is the total number of particles and $\hat{N}_j(n) = \psi_j^\dagger(n)\psi_j(n)$, $j = 1, 2$ are the particle number operators.

The L -operator satisfies the RTT -relation

$$R_{12}(u, v) L_1(u|n) L_2(v|n) = L_2(v|n) L_1(u|n) R_{12}(u, v)$$

$$R_{12}(u, v) = \mathbf{I} + g(u, v) \mathbf{P}_{12} \quad \text{with} \quad g(u, v) = \frac{-ic}{u - v}.$$

$\mathbf{P}_{12} \in \text{End}(\mathbb{C}^3) \otimes \text{End}(\mathbb{C}^3)$ is the permutation operator of the two auxiliary spaces 1 and 2. $R_{12}(u, v)$ is a $\text{GL}(3)$ -XXX R -matrix (associated to the Yangian $Y(gl_3)$).

The vacuum expectation values $\psi_j(n)|0\rangle_n = 0$, $j = 1, 2$, lead to

$$L(u|n) |0\rangle_n = \begin{pmatrix} 1 & 0 & \frac{-i\Delta\sqrt{c}}{1-iu\frac{\Delta}{2}} \psi_1^\dagger(n) \\ 0 & 1 & \frac{-i\Delta\sqrt{c}}{1-iu\frac{\Delta}{2}} \psi_2^\dagger(n) \\ 0 & 0 & r_0(u) \end{pmatrix} |0\rangle_n \quad \text{where} \quad r_0(u) = \frac{1 + \frac{iu\Delta}{2}}{1 - \frac{iu\Delta}{2}}. \quad (2)$$

The monodromy matrix reads $T(u) \equiv T_0(u|12 \dots M) = L_0(u|M) \cdots L_0(u|2) L_0(u|1)$, where M is the number of lattice sites. Since $T(u)$ obeys the RTT relation, the transfer matrix $t(u) = \text{tr}_0 T(u)$ defines an integrable model that includes in particular a lattice version of H_{NLS} .

In the representation associated to $|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_M$, we get:

$$T_{jj}(u)|0\rangle = \lambda_j(u) |0\rangle \quad \text{with} \quad \lambda_1(u) = \lambda_2(u) = 1 \quad \text{and} \quad \lambda_3(u) = (r_0(u))^M |0\rangle. \quad (3)$$

We get back to the continuous version through the limit: $\Delta \rightarrow 0$ and $M = \frac{L}{\Delta}$. In the continuous limit, $\lambda_3(u) \rightarrow e^{iuL}$, which implies:

$$\begin{aligned} T_{ij}(u) &\rightarrow \delta_{ij} + \frac{ic}{u} \int_0^L \psi_i^\dagger(y) \psi_j(y) dy + O(u^{-2}), & i, j = 1, 2, \\ T_{i3}(u) &\rightarrow -\frac{\sqrt{c}}{u} \left(e^{iuL} \psi_i^\dagger(L) - \psi_i^\dagger(0) \right) + O(u^{-2}), & i = 1, 2, \\ T_{3j}(u) &\rightarrow -\frac{\sqrt{c}}{u} \left(\psi_j(L) - e^{iuL} \psi_j^\dagger(0) \right) + O(u^{-2}), & j = 1, 2, \\ T_{33}(u) &\rightarrow e^{iuL} - \frac{ic}{u} e^{iuL} \int_0^L (\psi_1^\dagger(y) \psi_1(y) + \psi_2^\dagger(y) \psi_2(y)) dy + O(u^{-2}). \end{aligned}$$

Remark that $T(u)$ does not provide access to local operator, because $L(u)$ is not based on $R(u, v)$. Note also the 'unusual' behavior of $T(u)$ as $u \rightarrow \infty$ (w.r.t. 'usual' spin chains). Then, to compute form factors of local operators, we need to consider composite models and a modified version of the zero mode technics.

3 Notations

Besides the function $g(x, y) = \frac{-ic}{x - y}$ that enters in the definition of the R -matrix, we also introduce $f(x, y) = 1 + g(x, y)$.

Representations are labelled by the triplet $(\lambda_1(u), \lambda_2(u), \lambda_3(u))$. Associated to the representation (2) of the Lax operator, we have the functionals

$$r_1(u) = \frac{\lambda_1(u)}{\lambda_2(u)} \equiv 1 \rightarrow 1 \quad \text{and} \quad r_3(u) = \frac{\lambda_3(u)}{\lambda_2(u)} \equiv \left(\frac{1 - \frac{iu\Delta}{2}}{1 + \frac{iu\Delta}{2}} \right)^M \rightarrow e^{iuL}$$

where we have indicated (through \rightarrow) their value in the continuous limit.

We will use many sets of variables. The notation will be as follows:

- "bar" always denotes sets of variables: $\bar{w}, \bar{u}, \bar{v}$ etc..
- Individual elements of the sets have latin subscripts: w_j, u_k , etc..
- Subsets of variables are denoted by roman indices: $\bar{u}_I, \bar{v}_{IV}, \bar{w}_{II}$, etc.
- Special case: $\bar{u}_j = \bar{u} \setminus \{u_j\}$, $\bar{w}_k = \bar{w} \setminus \{w_k\}$, etc...

We use also shorthand notation for products of commuting operators / functions:

$$\begin{aligned} r_3(\bar{u}_{II}) &= \prod_{u_j \in \bar{u}_{II}} r_3(u_j); & T_{12}(\bar{u}) &= \prod_{u_j \in \bar{u}} T_{12}(u_j) \\ g(v_k, \bar{w}) &= \prod_{w_j \in \bar{w}} g(v_k, w_j); & f(\bar{u}_{II}, \bar{u}_I) &= \prod_{u_j \in \bar{u}_{II}} \prod_{u_k \in \bar{u}_I} f(u_j, u_k), \quad \text{etc..} \end{aligned}$$

4 Standard results from ABA

Explicit expressions for Bethe vectors. There are different expressions for BVs, see [1]. Here, we will focus on a particular one:

$$\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \sum \mathbb{K}_a(\bar{v}_I | \bar{u}) \frac{f(\bar{v}_I, \bar{v}_I)}{f(\bar{v}, \bar{u})} T_{13}(\bar{v}_I) T_{23}(\bar{v}_I) |0\rangle,$$

where $\mathbb{K}_a(\bar{v} | \bar{u})$ is the Izergin determinant.

The Bethe vectors are (right-)eigenvectors of the transfer matrix $t(w)$

$$t(w) \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \tau(w | \bar{u}; \bar{v}) \mathbb{B}^{a,b}(\bar{u}; \bar{v})$$

when the Bethe equations are obeyed (then, the vectors are called on-shell)

$$\left\{ \begin{array}{l} r_1(u_j) = \frac{f(u_j, \bar{u}_j)}{f(\bar{u}_j, u_j)} f(\bar{v}, u_j) \\ j = 1, 2, \dots, a \end{array} \right\}; \quad \left\{ \begin{array}{l} r_3(v_k) = \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)} f(v_k, \bar{u}) \\ k = 1, 2, \dots, b. \end{array} \right.$$

In the same way, dual Bethe vectors $\mathbb{C}^{a,b}(\bar{u}; \bar{v})$ are left eigenvectors (when on-shell)

$$\mathbb{C}^{a,b}(\bar{u}; \bar{v}) t(w) = \tau(w | \bar{u}; \bar{v}) \mathbb{C}^{a,b}(\bar{u}; \bar{v}).$$

Global form factors. The form factors for $T_{ij}(u)$ were computed in [6, 7]. For on-shell Bethe vectors $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ they have the following form:

$$\mathcal{F}_{a,b}^{(i,j)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{ij}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \quad (4)$$

$$= \left(\tau(z | \bar{u}^C, \bar{v}^C) - \tau(z | \bar{u}^B, \bar{v}^B) \right) F_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \quad (5)$$

$F_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$ is the *universal form factor*. It does not depend on the spectral parameter z , it is independent of the representation of $T(z)$ and admits a single determinant form for the GL(3) invariant R-matrix.

However, $T(u)$ provides acces only to global operators: to get form factors of local operators, we need to consider a more refined model, that we present now.

5 The composite model

The composite model framework was introduced in [9]. In the lattice version, we consider $T(u) = T^{(2)}(u|m) T^{(1)}(u|m)$, where

$$T^{(2)}(u|m) = L(u|M) \cdots L(u|m+1) \quad \text{and} \quad T^{(1)}(u|m) = L(u|m) \cdots L(u|1). \quad (6)$$

The integer $m \in [1, M[$ plays the role of the position x in the continuous version, and $T^{(j)}(u|m)$ are monodromy matrices for "shorter chains".

Vacuum expectation value. The vacuum has a factorized form $|0\rangle = |0\rangle^{(1)} |0\rangle^{(2)}$ so that

$$T_{jj}^{(1)}(u|m) |0\rangle^{(1)} = \ell_j(u) |0\rangle^{(1)}, \quad j = 1, 2, 3.$$

For the Bose gas model, $\ell_1(u) = \ell_2(u) = 1$ and

$$\ell_3(u) = \left(\frac{1 + \frac{i u \Delta}{2}}{1 - \frac{i u \Delta}{2}} \right)^m \rightarrow e^{i u x}.$$

Bethe vectors in composite model. [3, 4] An explicit expression is given by

$$\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{\ell_3(\bar{v}_{\text{II}})}{\ell_1(\bar{u}_{\text{I}})} f(\bar{u}_{\text{I}}, \bar{u}_{\text{II}}) f(\bar{v}_{\text{II}}, \bar{v}_{\text{I}}) f(\bar{v}_{\text{I}}, \bar{u}_{\text{I}}) \mathbb{B}_{a_{\text{I}}, b_{\text{I}}}^{(1)}(\bar{u}_{\text{I}}; \bar{v}_{\text{I}}) \mathbb{B}_{a_{\text{II}}, b_{\text{II}}}^{(2)}(\bar{u}_{\text{II}}; \bar{v}_{\text{II}}),$$

where $\mathbb{B}_{a_{\text{I}}, b_{\text{I}}}^{(j)}(\bar{u}; \bar{v})$ are Bethe vectors for the partial monodromy matrices $T^{(j)}(u|m)$, $j = 1, 2$.

Note that the 'partial' BVs $\mathbb{B}_{a_{\text{I}}, b_{\text{I}}}^{(j)}(\bar{u}; \bar{v})$, $j = 1, 2$, are off-shell even when the total BV $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$ is on-shell.

Local form factors. [5] We are interested in computing $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{ij}^{(1)}(u|m) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$, where the Bethe vectors are on-shell. It is a local form factor because of the position m .

More precisely, we want to compute

$$\mathbf{M}_{a,b}^{(i,j)}(x|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{ij}^{(1)}[0] \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$$

where $T_{ij}^{(1)}[0]$ is the zero mode (the Lie algebra part) of $T_{ij}^{(1)}(u|m)$, that we detail below.

6 The zero mode method

The zero modes for the Bose gas are defined as

$$\begin{aligned} T_{ij}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (T_{ij}(u) - \delta_{ij}), \quad i, j = 1, 2 & T_{33}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (e^{-iLu} T_{33}(u) - 1), \\ T_{j3}[0] &= \lim_{u \rightarrow -i\infty} \frac{u}{c} e^{-iLu} T_{j3}(u), & T_{3j}[0] &= \lim_{u \rightarrow i\infty} \frac{u}{c} T_{j3}(u), \quad j = 1, 2 \end{aligned}$$

They generate an $SL(3)$ Lie algebra:

$$\begin{aligned} [T_{ij}[0], T_{kl}[0]] &= \delta_{i,l} T_{kj}[0] - \delta_{j,k} T_{il}[0], \quad i, j, k, l = 1, 2, 3, \\ [T_{ij}[0], T_{kl}(z)] &= \delta_{i,l} T_{kj}(z) - \delta_{j,k} T_{il}(z), \quad i, j, k, l = 1, 2, 3. \end{aligned}$$

In particular $[T_{ij}[0], t(z)] = 0$: they are a symmetry of the model.

Bethe vectors and zero modes. Zero modes appear naturally in the expression of BVs:

$$\begin{aligned} T_{12}[0] \mathbb{B}^{a,b}(\bar{u}; \bar{v}) &= \lim_{|w| \rightarrow \infty} \frac{w}{c} \mathbb{B}^{a+1,b}(\{w, \bar{u}\}; \bar{v}), \\ T_{23}[0] \mathbb{B}^{a,b}(\bar{u}; \bar{v}) &= \lim_{w \rightarrow -i\infty} e^{-iwL} \frac{w}{c} \mathbb{B}^{a,b+1}(\bar{u}; \{w, \bar{v}\}), \end{aligned}$$

with similar expressions for dual Bethe vectors. Note that the limit $|w| \rightarrow \infty$ preserves the Bethe equations, i.e. the Bethe vector stays on-shell.

Moreover, for on-shell Bethe vectors:

$$\begin{aligned} T_{21}[0] \mathbb{B}^{a,b}(\bar{u}; \bar{v}) &= T_{32}[0] \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = T_{31}[0] \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = 0 \\ \mathbb{C}^{a,b}(\bar{u}; \bar{v}) T_{12}[0] &= \mathbb{C}^{a,b}(\bar{u}; \bar{v}) T_{23}[0] = \mathbb{C}^{a,b}(\bar{u}; \bar{v}) T_{13}[0] = 0, \end{aligned}$$

where the Bethe parameters \bar{u} and \bar{v} are supposed to be finite.

The zero mode method is a way to compute form factors using zero modes [2]. The main idea is to use the Lie algebra symmetry generated by the zero modes and the highest weight property of (on-shell) Bethe vectors to obtain relations among the form factors (4). As an example, let us show how to relate $\mathcal{F}_{a+1,b}^{(2,2)}(z|\bar{u}^C, \bar{v}^C; \{\bar{u}^B, w\}, \bar{v}^B)$ to $\mathcal{F}_{a,b}^{(1,2)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$:

$$\begin{aligned} \lim_{w \rightarrow \infty} \frac{w}{c} \mathcal{F}_{a+1,b}^{(2,2)}(z|\bar{u}^C, \bar{v}^C; \{\bar{u}^B, w\}, \bar{v}^B) &= \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{22}(z) \lim_{w \rightarrow \infty} \frac{w}{c} \mathbb{B}^{a+1,b}(\{\bar{u}^B, w\}; \bar{v}^B) \\ &= \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{22}(z) T_{12}[0] \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) [T_{22}(z), T_{12}[0]] \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \\ &= \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{12}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \mathcal{F}_{a,b}^{(1,2)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \end{aligned}$$

More examples of such relations can be found in [2].

Zero mode in the composite model. We can defined 'local' zero modes, through the composite model:

$$\begin{aligned} T_{ij}^{(1)}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (T_{ij}^{(1)}(u|x) - \delta_{ij}) = - \int_0^x \psi_i^\dagger(y) \psi_j(y) dy, \quad i, j = 1, 2 \\ T_{j3}^{(1)}[0] &= \lim_{u \rightarrow -i\infty} \frac{u}{c} e^{-ixu} T_{j3}^{(1)}(u|x) = -\frac{1}{\sqrt{c}} \psi_j(x), \quad j = 1, 2 \\ T_{3j}^{(1)}[0] &= \lim_{u \rightarrow i\infty} \frac{u}{c} T_{3j}^{(1)}(u|x) = -\frac{1}{\sqrt{c}} \psi_j^\dagger(x), \quad j = 1, 2 \\ [T_{ij}^{(1)}[0], T_{kl}^{(1)}[0]] &= \delta_{i,l} T_{kj}^{(1)}[0] - \delta_{j,k} T_{il}^{(1)}[0], \quad i, j, k, l = 1, 2, 3. \end{aligned}$$

The zero mode method applied to local zero modes allows to relate the different local form factors $\mathbf{M}_{a,b}^{(i,j)}(x|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$. Then, we need to compute only one local form factor. It is done using the twisted transfer matrix method.

7 The twisted transfer matrix method

Still associated to the monodromy matrix $T(u)$, one can introduce a twisted version of the transfer matrix

$$t_{\bar{\beta}}(z) = \sum_{j=1}^3 e^{\beta_j} T_{jj}(z) = \text{tr}_0(M_{\bar{\beta}} T(u)), \quad M_{\bar{\beta}} = \text{diag}(e^{\beta_1}, e^{\beta_2}, e^{\beta_3}), \quad \beta_j \in \mathbb{C}$$

that is integrable. Twisted BVs (constructed from ABA) can be associated to this twisted transfer matrix. They are eigenvectors of $t_{\bar{\beta}}(z)$

$$t_{\bar{\beta}}(w) \mathbb{B}_{\bar{\beta}}^{a,b}(\bar{u}; \bar{v}) = \tau_{\bar{\beta}}(w|\bar{u}; \bar{v}) \mathbb{B}_{\bar{\beta}}^{a,b}(\bar{u}; \bar{v})$$

when the Bethe equations are obeyed

$$r_1(u_j) = e^{\beta_1 - \beta_2} \frac{f(u_j, \bar{u}_j)}{f(\bar{u}_j, u_j)} f(\bar{v}, u_j), \quad r_3(v_k) = e^{\beta_2 - \beta_3} \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)} f(v_k, \bar{u}).$$

Accordingly, the twisted scalar product $\mathcal{S}_{(\bar{\beta})}^{a,b} = \mathbb{C}_{\bar{\beta}}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}_{\bar{\beta}}^{a,b}(\bar{u}^B; \bar{v}^B)$ can be computed. It is known as a single determinant [6].

The twisted transfer matrix method allows to compute diagonal form factors for the untwisted transfer matrix, starting from scalar products associated to the twisted one. The generating functional for such form factors is given by

$$G_{(\bar{\beta})}^{a,b} = \mathbb{C}_{(\bar{\beta})}^{a,b}(\bar{u}^C; \bar{v}^C) e^{Q_{\bar{\beta}}} \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = e^{\beta_1 \ell_1[0] + \beta_3 \ell_3[0]} \frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{v}^C)} \mathcal{S}_{(\bar{\beta})}^{a,b}$$

$$Q_{\bar{\beta}} = \sum_{j=1}^3 \beta_j T_{jj}^{(1)}[0], \quad \beta_j \in \mathbb{C}.$$

The generating functional allows to compute the diagonal form factor

$$\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{ii}^{(1)}[0] \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \left(\frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{v}^C)} - 1 \right) \frac{d}{d\beta_i} \mathcal{S}_{(\bar{\beta})}^{a,b} \Big|_{\bar{\beta}=0}.$$

8 Results

Gathering the different results exposed above, one can compute the desired form factors [8]:

$$\mathbb{C}_{a,b}(\bar{u}; \bar{v}) \psi_j^\dagger(x) \psi_j(x) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = i \sum_{k=1}^b \frac{dv_k(\bar{\beta})}{d\beta_j} \Big|_{\bar{\beta}=0} \|\mathbb{B}_{a,b}(\bar{u}; \bar{v})\|^2, \quad j = 1, 2,$$

$$\begin{aligned} \mathbb{C}_{a',b}(\bar{u}^C; \bar{v}^C) \psi_i^\dagger(x) \psi_j(x) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) &= -i \mathcal{P}(\bar{v}^B, \bar{v}^C) e^{ix\mathcal{P}(\bar{v}^B, \bar{v}^C)} \mathbf{F}_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \\ a' = a + j - i \quad \{\bar{u}^C; \bar{v}^C\} &\neq \{\bar{u}^B; \bar{v}^B\} \\ \mathbb{C}_{a-2+k, b-1}(\bar{u}^C; \bar{v}^C) \psi_k(x) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) &= i\sqrt{c} e^{ix\mathcal{P}(\bar{v}^B, \bar{v}^C)} \mathbf{F}_{a,b}^{(3,k)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \\ \mathbb{C}_{a+2-k, b+1}(\bar{u}^C; \bar{v}^C) \psi_k^\dagger(x) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) &= i\sqrt{c} e^{ix\mathcal{P}(\bar{v}^B, \bar{v}^C)} \mathbf{F}_{a,b}^{(k,3)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \end{aligned}$$

Above, $\mathbf{F}_{a,b}^{(k,l)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$ are the universal form factors (that are known, see [2]), and

$$\mathcal{P}(\bar{v}^B, \bar{v}^C) = \sum_{i=1}^b v_i^B - \sum_{i=1}^{b'} v_i^C.$$

9 Perspectives

The present results will be used to calculate the mean values $\langle \psi_j^\dagger(x) \psi_j(x) \rangle$, $\langle \psi_j^\dagger(x) \rangle$ and $\langle \psi_j(x) \rangle$, for applications to condensed matter experiments.

We also obtained a conjecture for the form factors of local operator in $gl(N)$ composite models:

$$\begin{aligned} \mathbb{C}_{\bar{b}}(\bar{s}) T_{ij}^{(1)}[0] \mathbb{B}_{\bar{a}}(\bar{t}) &= \left(\prod_{k=1}^{N-1} \frac{\alpha_k(\bar{s}^k)}{\alpha_k(\bar{t}^k)} - 1 \right) \mathbf{F}_{\bar{a}}^{(i,j)}(\bar{s}; \bar{t}), \quad \text{for } \bar{s} \neq \bar{t} \\ \mathbb{C}_{\bar{a}}(\bar{t}) T_{jj}^{(1)}[0] \mathbb{B}_{\bar{a}}(\bar{t}) &= \left(\lambda_j^{(1)}[0] + \sum_{k=1}^{N-1} \frac{d}{d\kappa_j} \log \alpha_k(\bar{t}^k(\bar{\kappa})) \Big|_{\bar{\kappa}=1} \right) \|\mathbb{B}_{\bar{a}}(\bar{t})\|^2. \end{aligned}$$

with $\alpha_j(u) = \frac{\lambda_j^{(1)}(u)}{\lambda_{j+1}^{(1)}(u)}$; $T_{jj}^{(1)}(u) |0\rangle^{(1)} = \lambda_j^{(1)}(u) |0\rangle^{(1)}$, $j = 1, \dots, N-1$. Note however that

we still need a determinant form for the scalar product and/or $F_a^{(i,j)}(\bar{s}; \bar{t})$.

Finally, the superalgebra case is investigated: work is in progress for the rational R -matrix, for the computation of the Bethe vectors, scalar products and form factors of the model.

References

- [1] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., *Bethe vectors of $GL(3)$ -invariant integrable models*, J. Stat. Mech. **1302** (2013) P02020, [arXiv:1210.0768](#).
- [2] Pakuliak S., Ragoucy E., Slavnov N.A., *Zero modes method and form factors in quantum integrable models*, Nucl. Phys. **B893** (2015) 459-481, [arXiv:1412.6037](#).
- [3] Slavnov N.A., *One-dimensional two-component Bose gas and the algebraic Bethe ansatz*, [arXiv:1502.06749](#).
- [4] Pakuliak S., Ragoucy E., Slavnov N.A., *$GL(3)$ -based quantum integrable composite models: 1. Bethe vectors*, SIGMA **11** (2015) 063, [arXiv:1501.07566](#).
- [5] Pakuliak S., Ragoucy E., Slavnov N.A., *$GL(3)$ -based quantum integrable composite models: 2. Form factors of local operators*, SIGMA **11** (2015) 064, [arXiv:1502.01966](#).
- [6] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., *Form factors in $SU(3)$ -invariant integrable models*, J. Stat. Mech. **1309** (2013) P04033, [arXiv:1211.3968](#);
- [7] Pakuliak S., Ragoucy E., Slavnov N.A., *Determinant representations for form factors in quantum integrable models with $GL(3)$ -invariant R -matrix*, Theor. Math. Phys. **181** (2014) 1566-1584, [arXiv:1406.5125](#);
Pakuliak S., Ragoucy E., Slavnov N.A., *Form factors in quantum integrable models with $GL(3)$ -invariant R -matrix*, Nucl. Phys. **B881** (2014) 343-368, [arXiv:1312.1488](#).
- [8] Pakuliak S., Ragoucy E., Slavnov N.A., *Form factors of local operators in a one-dimensional two-component Bose gas*, J. Phys. **A48** (2015) 435001, [arXiv:1503.00546](#).
- [9] Izergin A.G., Korepin V. E., *The quantum inverse scattering method approach to correlation functions*, Comm. Math. Phys. **94** (1984) 67–92.
- [10] Kulish P. P., Reshetikhin N. Yu., *$GL(3)$ -invariant solutions of the Yang-Baxter equation and associated quantum systems*, Zap. Nauchn. Sem. POMI. **120** (1982) 92–121; J. Sov. Math. **34:5** (1982) 1948–1971 (Engl. transl.).